0.1 THE NUMBER OF INTEGRAL DISTANCE POINTS ON A CIRCLE

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AbstractPoints are called integral if all the distances between them are integers. The paper finds the number of integral distance points on any given circle of integer diameter and asserts that if the number of multiples constructed of $4k+1$ prime factors of integer d equals n, then a circle with diameter d contains maximum of $2n$ points such that distances between any pair of points are integers and that the circle cannot contain $2n+1$ such points.

1. Literature. Numbers and positions of points with integral distances on a plane has long been an interesting subject. Before 1945, S.Ulam conjectured that a set of densely populated points with rational distances does not exist [1] but the conjecture has yet to be proven [2]. In 1945, N.Anning, P.Erdцs [3] made considerable advances on the subject. Some of those are:

Theorem 1. If an infinitely many points in the plane have distances that are all integrals, then they all exist on a line.

Theorem 2. For any n we can find n points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line).

The proof of Theorem 2 requires three different constructions [3] as follows:

A. Consider the circle of diameter 1, $x^2+y^2=\frac{1}{4}$. The sequence of $4k+1$ prime numbers p_1, p_2, ... is to be used in the configuration. According to Fermat's theorem on sums of two squares, $p_i^2=a_i^2+b_i^2$, $a_i\neq 0$, $b_i\neq 0$ is solvable. Applying this yields a construction of sequence of (x_i,y_i) points on the circle $x^2+y^2=\frac{1}{4}$ that are the verticis of right triangles with catheti of $\dfrac{a_i}{p_i},\dfrac{b_i}{p_i}$ respectively. The resulting points (x_i,y_i) can easily be verified to have rational distances from one another. Thus of course by enlarging the radius of the circle, more points with properties such that all have integral distances from one another. It is also conjectured that these points are dense on the circle.

B. Consider the point $(0, \frac{1}{2})$ of the circle $x^2+y^2=\frac{1}{4}$. If the point is rotated by the center by integer repetitions of angle $\alpha = \arccos\frac{4}{5}$, then infinitely many different configurations of points with rational distances and densely populated.

C. Let m^2 be an odd number with d divisors and consider the equation $m^2 = x^2 - y^2$. This equation clearly has exactly d solutions. It is immediate that all the distances between points $(m, 0)$, $(0, y_1)$, ... , $(0, y_d)$ are, in fact, integral. Before this accomplished result of Erdцs-Anning, the author proposed and selected the following related problems to the Mongolian Mathematical Olympiad (MMO) [4].

Problem 1, (B.Bayasgalan 1988). For which values of integer D, there exist 3 integral distance points on a circle of diameter D?

Problem 2, (B.Bayasgalan MMO-1988). a) Prove that there exist not, on a circle of diameter 7^n, 3 points with integer distances from each other.

b) Prove that there exist, on a circle of diameter 5^n, $2n+2$ points with integer distances from each other.

Furthermore, the following problem was proposed and tried at the IMO-2014.

Core problem, (B.Bayasgalan, 1993). Find the maximum number of points on a circle of diameter D such that each distance between any two points is an integer where D is an integer.

2. 2. Core result. The following problem was proposed and tried in the selection process of the Mongolian team for the IMO.

Problem 3, (B.Bayasgalan, B.Bayarjargal). Prove that the sufficient and necessary condition for finding four integral distance points on a circle of diameter integer D is that D is divisible by at least one $4k+1$ prime number.

Additionally, B.Bayasgalan, on behalf of the Mongolian team, proposed to the IMO the following problem.

Problem 4. Prove that there exists, on a circle of diameter D, $2n+2$ number of points such that any two points have integer distance, where d is an integer with n number of $4k+1$ prime factors. Theorem (On the number of integral points on a cicle with integer diameter).

Theorem (On the number of integral points on a cicle with integer diameter). Let $D = D' \cdot p_1^{s_1} \cdot ... \cdot p_k^{s_k}$, where D' is indivisible by prime numbers in the form of $4s+1$, $p_1, ..., p_k$ are prime numbers in the form of $4s+1$, and $P(D) = (s_1+1)(s_2+1)...(s_k+1)$. Then maximum of $2P(D)$ points can placed on a circle of diameter of D such distances between any two points are integers.

For instance: 1) If $D = 7^n$, then $P(D) = 1$. Thus there exist not 3 points on a circle of diameter D such that any two of the points have integer distance between them.

2) If $D = 5^n$, $P(D) = n+1$. then Thus there exist $2(n+1)$ number of points on a circle of diameter D such that any two of the points have integer distance between them. But there cannot be $2n+3$ such points on the circle.

3) If there exist 5 points on a circle of diameter D integer such that any two of the points have integer distance between them, then it is possible to find 6 such points and that D has at least two $4s+1$ prime factors ($P(D) \geq 2$).

Proof (B.Ganbileg, B.Bayasgalan). 1) Suppose there exist at least 3 points on the circle of diameter D such that any two points have integer distance between them. Also, let $D = D' \cdot p_1 p_2 .... p_k$, where p_i is a $4s+1$ prime number and D' is not divisible by $4s+1$ prime numbers. Then let's prove that the distances between these points are, in fact, divisible by D'. From figure 1

$a = D \cdot \sin\alpha$, $b = D \cdot \sin\beta$, $c = D \cdot (\sin\alpha\cos\beta + \sin\beta\cos\alpha)$ From this, considering $\sin\alpha = \dfrac{m}{n}$, $\sin\beta = \dfrac{p}{q} \Rightarrow \dfrac{m}{n}\cos\beta + \dfrac{p}{q}\cos\alpha \in \mathbb{Q}$, where $(m,n) = (p,q) = 1$.

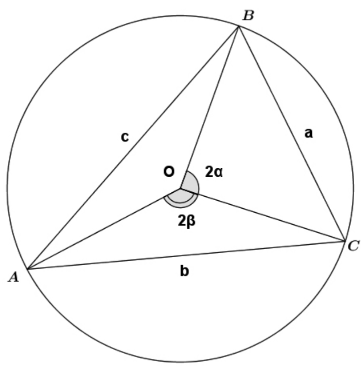

Figure 1

$$\frac{m}{n}\sqrt{1-\frac{p^2}{q^2}}+\frac{p}{q}\sqrt{1-\frac{m^2}{n^2}}=\frac{x}{y} \qquad x,y,m,n,p,q\in\mathbb{N}$$

Thus, $\sqrt{q^2-p^2}$, $\sqrt{n^2-m^2}\in\mathbb{Z}$.

In other words, $(\sqrt{n^2-m^2},m,n)$, $\sqrt{q^2-p^2},p,q$ comprise primitive Pythagorean triples. The long side of the Pythagorean triple is an odd number and is not divisible by $4s+3$ prime factors is derived directly from Fermat's theorem on sums of two squares. Therefore, distances are divisible by a,b,c.

2) Let $D=p_1^{s_1}p_2^{s_2}...p_k^{s_k}$ and p_i is a $4s+1$ prime number. Now consider angles $\alpha_1,...,\alpha_k$ such that $\cos\alpha_i=\dfrac{a_i}{p_i}$, $\sin\alpha_i=\dfrac{b_i}{p_i}$. Such natural numbers a_i,b_i exist uniquely (Fermat's theorem on sums of two squares).

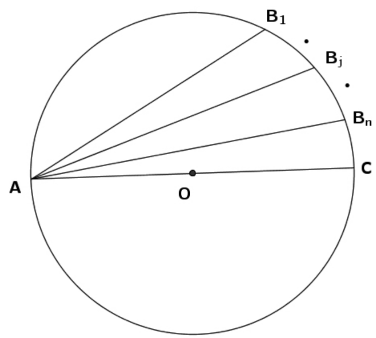

Figure 2

Configure angles $\sum_{i=1}^{k}t_i\alpha_i$ $(0\leq t_i\leq s_i)$ on the semicircle as shown in Figure 2. If an angle $\sum t_i\alpha_i$ is greater then π, then substract from then angle πs to for configuration. Here, total number of $P(D)=(s_1+1)(s_2+1)...(s_k+1)$ points have been constructed and each point constitutes vertex $\angle B_jAC$. Difference between any two of the angles is in the form of $\sum_{i=1}^{k}r_i\alpha_i$ $(-s_i\leq r_i\leq s_i)$. In other words, the distance between any two configured points is an integer. The total number of points, together with the points on the opposite side to the diameter, is $2P(D)$.

The following is to ensure that the configurated points are distinct. Suppose

$\sum\limits_{i=1}^{k} 2t_i\alpha_i = 2\pi s$. Then $\prod\limits_{j=1}^{k} (\cos 2\alpha_j + i \sin 2\alpha_j)^{t_j} = 1$ and

$$\prod_{j=1}^{k} \left(\frac{a_j}{p_j} + i\frac{b_j}{p_j}\right)^{2t_j} = \frac{\prod\limits_{j=1}^{k} (a_j + ib_j)^{2t_j}}{\prod\limits_{j=1}^{k} (a_j + ib_j)^{t_j} (a_j - ib_j)^{t_j}} = 1 \tag{1}$$

This yields the contradiction that the Gaussian integer/prime factorization equals one.

3) Now let's prove that the number of integral distance points on the circle of diameter interger D does not exceed $2P(D)$.

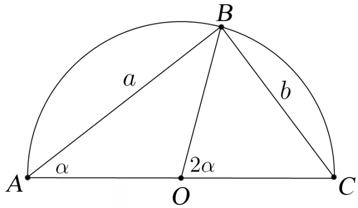

Figure 3

Let $D = p_1^{s_1}...p_k^{s_k}$, and $p_1, ..., p_k$ are $4s+1$ distinct prime numbers. Consider the point B on a semicircle of diameter D(Figure 3). If B is a complex number, then $B = \frac{D}{2}e^{2\alpha i}$ and if $\cos\alpha = \frac{a}{D}$, $\sin\alpha = \frac{b}{D}$, then $a^2 + b^2 = D^2$.

On the other hand, note that $p_j = a_j^2 + b_j^2$ and $a_j + ib_j$, $a_j - ib_j$ are Guassian prime numbers. Then, it follows that

$$a + ib = \prod_{1 \le j \le k} (a_j + ib_j)^{t_j} \cdot (a_j - ib_j)^{2s_j - t_j}.$$

Here, $0 \le t_j \le 2s_j$. Therefore

$$B = \frac{D}{2}\left(\frac{a}{D} + i\frac{b}{D}\right)^2 = \frac{1}{2D}(a+ib)^2 = \frac{1}{2D}\prod_{1 \le j \le k} (a_j + ib_j)^{2t_j} (a_j - ib_j)^{4s_j - 2t_j} =$$

$$= \frac{1}{2D}\prod_{1 \le j \le k} p_j^{2s_j} \cdot e^{i2t_j\varepsilon_j} \cdot e^{-i(4s_j - 2t_j)\varepsilon_j}, \ \varepsilon_j = \arccos\frac{a_j}{\sqrt{p}_j}.$$

Therefore, $B = \frac{D}{2}\prod\limits_{1 \le j \le k} e^{4i(t_j - s_j)\varepsilon_j}$. For any other point, it follows that

$B' = \frac{D}{2} \prod_{1 \le j \le k} e^{4i(t'_j - s_j)\varepsilon_j}$. Thus,

$$BB' = \left| D \cdot \sin 2 \sum_{1 \le j \le k} (t_j - t'_j)\varepsilon_j \right| = D \cdot \frac{A}{\prod_{1 \le j \le k} p_j^{|t_j - t'_j|}}.$$

The numerator A and the denominator are relatively prime, or coprime. It follows that $|t_j - t'_j| \le s_j$ or $-s_j \le t_j - t'_j \le s_j$. Therefore, the number of distinct values of t_j is at most $s_j + 1$. Hence, the number such points do not exceed $2P(D)$. Theorem proven.

Corollary. The maximum number of integral points on a circle of integer diameter can be configured via rotation and symmetrical approximation only once.

3. Conclusion. The main result above is a general case solution to Problems 1, 2, 3, and 4. Authors do not claim to know any comparable or equivalent results that have been formulated up to now. In such case of similarity or repetition, authors express independence. This finding is dedicated to Ts.Dashdorj whose contribution continues to be a driving force of advancements in Mongolian mathematics.

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